Review

# Recent Developments in Eddy Viscosity Modelling of Turbulence

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## Abstract

Eddy viscosity modelling is still a "standard" approach for industrial CFD applications for turbulent flows despite its serious deficiencies. Thus, a number of research studies, including the author's, have been recently made to improve models of this kind. This article reviews these efforts and suggests a future direction for tackling turbulent flows of industrial importance.

Keywords: CFD, Modelling turbulence, Eddy viscosity, Near-wall model, Nonlinear stress-strain relation

## 1. Introduction

Turbulence is an irregular motion in fluid flows. The various flow quantities thus show random variation with time and space and only statistically averaged values are distinctively discerned. Since exactly dealing with turbulence with mathematics has been one of the most notoriously thorny problems of classical physics, several well known scientists, who had interests in turbulence, did not dare to pursue its physics. For example, W. Heisenberg, the great Nobel Prize laureate, did his doctoral research on turbulent flows but later changed his major to quantum theory. A. Einstein, the greatest Nobel Prize physicist in this century, knew the difficulty of turbulence and thus did not go into its physics. R. Feynman, another famous Nobel Prize laureate, noted that turbulence was the most important unsolved problem of classical physics.

However, the development of modern computer technology drastically changed this situation and has provided some opportunities even for industrial engineers to challenge turbulent flows using Computational Fluid Dynamics (CFD). The major CFD treatments of turbulence can be classified into four types of approaches: Eddy Viscosity Model (EVM), Reynolds Stress Model (RSM), Large Eddy Simulation (LES) and Direct Numerical Simulation (DNS). Amongst these, only DNS closely simulates the actual physics of turbulence, while *industrially* interesting schemes are at present the simplest EVM's because the others require far more computer resources than are available for routine work.

Although the eddy viscosity concept assumes a crude relation between turbulent quantities, models based on this concept such as the $k$-$\varepsilon$ two-equation model have made many successful predictions in many flow fields with numerical stability. The $k$-$\varepsilon$ EVM is thus the main scheme for routine work in present industrial laboratories. Nevertheless, over the wide range of flow predictions made over the last two decades, it is now recognized that conventional EVM's have severe defects in many complicated flow fields. Consequently, research studies aimed at extending the applicability of EVM's have been highly demanded by industry. In fact, the continuous research efforts have been significantly extending the performance of EVM's. Therefore, this article particularly focuses on these recent achievements.

The following section §2 surveys the main historical establishments related to eddy viscosity modelling, section §3 summarizes the recent efforts, then finally, section §4 concludes and suggests a future direction for the treatment of industrially important turbulent flows.

## 2. Historical foundations

In this section, firstly, major establishments in eddy viscosity modelling are surveyed, then a more extended modelling concept that forms algebraic Reynolds stress models (ASM's) is summarized because it is mathematically very close to EVM's and links some recent nonlinear eddy viscosity approaches.

### 2. 1 Eddy viscosity models

The EVM's are based on an algebraic expression which represents the Reynolds stresses appearing in the ensemble averaged Navier-Stokes equations as unknown properties. The ensemble averaged forms of the transport equations for mean velocity of incompressible flows with constant properties can be written as:

Continuity,

$$\frac{\partial U_i}{\partial x_i} = 0 \qquad \cdots\cdots(1)$$

Momentum,

$$\frac{DU_i}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial U_i}{\partial x_j}\right) - \frac{\partial}{\partial x_j}(\underbrace{\overline{u_i u_j}}_{\text{Reynolds stress}}) \qquad \cdots\cdots(2)$$

where $\rho$ and $\nu$ are, respectively, the density and the kinematic viscosity of the fluid of interest. Following Boussinesq[1], the Reynolds stresses are represented by the eddy viscosity $\nu_t$ and the strain tensor $S_{ij}$ as:

$$\overline{u_i u_j} = \frac{2}{3}k\delta_{ij} - \nu_t S_{ij}. \qquad \cdots\cdots(3)$$

The strain tensor is defined as:

$$S_{ij} \equiv \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right). \qquad \cdots\cdots(4)$$

The $k$-$\varepsilon$ EVM[2] takes the eddy viscosity as:

$$\nu_t = c_\mu \frac{k^2}{\varepsilon} \qquad \cdots\cdots(5)$$

where $c_\mu$ is a constant value given by referring to local equilibrium shear layers. To obtain $\nu_t$, the transport equations for the turbulent kinetic energy $k$ and its dissipation rate $\varepsilon$ are solved with approximations.

$$\frac{Dk}{Dt} = D_k \underbrace{- \overline{u_i u_j}\frac{\partial U_i}{\partial x_j}}_{P_k} \underbrace{- \nu\overline{\frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_i}}}_{\varepsilon} \qquad \cdots\cdots(6)$$

$$\frac{D\varepsilon}{Dt} = D_\varepsilon + \frac{c_{\varepsilon 1}P_k - c_{\varepsilon 2}\varepsilon}{k/\varepsilon} \qquad \cdots\cdots(7)$$

where $D_\phi$ is the diffusive transport term of the variable $\phi$ which is normally modelled as:

$$D_\phi = \frac{\partial}{\partial x_j}\left\{\left(\nu + \frac{\nu_t}{\sigma_\phi}\right)\frac{\partial \phi}{\partial x_j}\right\}. \qquad \cdots\cdots(8)$$

The coefficients $c_{\varepsilon 1}$ and $c_{\varepsilon 2}$ were given by referring to the measured rate of decay in grid turbulence and the local equilibrium turbulence, respectively. These standard values[3] are listed on **Table 1**. This standard version does not have any near-wall dependence upon molecular viscosity, so that *wall functions*[4,5] are employed in place of the no-slip wall boundary condition.

To take account of the viscous effects, Jones & Launder[2] (JL) first devised a low-Reynolds-number (LRN) version of the $k$-$\varepsilon$ EVM. They implemented the near-wall viscous effects by damping $\nu_t$ toward a wall with introduction of a *damping function*, $f_\mu$, as:

$$\nu_t = c_\mu f_\mu \frac{k^2}{\tilde{\varepsilon}}. \qquad \cdots\cdots(9)$$

This $f_\mu$ function was designed to reduce its value from unity toward a wall. They also modified the $\varepsilon$ equation with the introduction of the other damping functions, $f_1$ and $f_2$, as:

$$\frac{D\tilde{\varepsilon}}{Dt} = D_{\tilde{\varepsilon}} + \frac{c_{\varepsilon 1}f_1 P_k - c_{\varepsilon 2}f_2\tilde{\varepsilon}}{k/\tilde{\varepsilon}} + P_{\varepsilon 3} \qquad \cdots\cdots(10)$$

where $\tilde{\varepsilon}$ is the isotropic part of $\varepsilon$ defined as $\tilde{\varepsilon} \equiv \varepsilon - 2\nu\left(\frac{\partial k^{1/2}}{\partial x_j}\right)^2$ . The reason why they chose $\tilde{\varepsilon}$ rather than $\varepsilon$ itself is that $\tilde{\varepsilon}$ vanishes to exactly 0 at a wall boundary. This simple boundary condition makes numerical solutions more stable. The

**Table 1** Empirical constants for the $k$-$\varepsilon$ EVM.

| $c_\mu$ | $c_{\varepsilon 1}$ | $c_{\varepsilon 2}$ | $\sigma_k$ | $\sigma_\varepsilon$ |
|---|---|---|---|---|
| 0.09 | 1.44 | 1.92 | 1.0 | 1.3 |

gradient production term, $P_{\varepsilon 3}$, was also modelled using a gradient diffusion hypothesis as:

$$P_{\varepsilon 3} = 2\nu\nu_t \left( \frac{\partial^2 U_i}{\partial x_j \partial x_k} \right)^2 . \quad \cdots\cdots (11)$$

Many versions followed this original work. Some researchers such as Wilcox[6], for example, chose a substitute quantity for $\varepsilon$. This model solves a modelled equation for $\omega$ ( $\equiv \varepsilon / k$ ):

$$\frac{D\omega}{Dt} = D_\omega + c_{\omega 1} P_k \frac{\omega}{k} - c_{\omega 2}\omega^2 . \quad \cdots\cdots (12)$$

The coefficients such as $c_{\omega 1}$ and $c_{\omega 2}$ were tuned by referring to the $k$-$\varepsilon$ model equations since the $\omega$ equation was derived by manipulating the $k$ and $\varepsilon$ equations.

Patel *et al.*[7] concluded, however, following systematic comparisons between eight LRN models, that an amended version of the JL model by Launder & Sharma[8] (LS) was one of the most successful for a number of straight thin shear flows. The LS model uses the following damping functions of a turbulent Reynolds number $R_t$ ( $\equiv k^2 / ( \nu\tilde{\varepsilon}$ ).

$$\begin{cases} f_\mu = \exp\left\{ \dfrac{-3.4}{(1 + R_t / 50)^2} \right\} \\ f_1 = 1.0 \\ f_2 = 1.0 - 0.3\exp(-R_t^2) \end{cases} \quad \cdots\cdots (13)$$

### 2. 2 Algebraic Reynolds Stress Models

The transport equation of the Reynolds stress $\overline{u_i u_j}$ is

$$\frac{D\overline{u_i u_j}}{Dt} = D_{ij} - \underbrace{\left( \overline{u_i u_k}\frac{\partial U_j}{\partial x_k} + \overline{u_j u_k}\frac{\partial U_i}{\partial x_k} \right)}_{P_{ij}} + \Pi_{ij} - \varepsilon_{ij} \quad \cdots\cdots (14)$$

where $\Pi_{ij}$ and $\varepsilon_{ij}$ are, respectively, the pressure correlation and the dissipation rate terms of $\overline{u_i u_j}$. The terms that contain the gradient of $\overline{u_i u_j}$ are the transport terms (*i.e.,* the convection and the diffusion terms). In the ASM scheme, the transport of the Reynolds stresses is approximated in terms of that of turbulence energy $k$ to reduce the differential equations for $\overline{u_i u_j}$ to a set of algebraic ones. This scheme was firstly introduced by Rodi[9] as:

$$T_{ij} = T_k \frac{\overline{u_i u_j}}{k} \quad \cdots\cdots (15)$$

where $T_\phi$ is namely the *net* transport (convection minus diffusion) of $\phi$. Since $T_k = P_k - \varepsilon$, by adapting Eq.(15), one may rewrite Eq.(14) as:

$$P_{ij} + \Pi_{ij} - \varepsilon_{ij} = \frac{\overline{u_i u_j}}{k} ( P_k - \varepsilon ) \quad \cdots\cdots (16)$$

Then, one just needs models for the terms $\Pi_{ij}$ and $\varepsilon_{ij}$. There are many established models for $\Pi_{ij}$ such as Launder, Reece & Rodi model[10], and each model forms a different version of the ASM. However, the basic model[11] may be

$$\Pi_{ij} = -c_1 \varepsilon a_{ij} - c_2 \left( P_{ij} - \frac{1}{3}\delta_{ij} P_{kk} \right) \quad \cdots\cdots (17)$$

where the anisotropic stress tensor $a_{ij} = \overline{u_i u_j} / k - \frac{2}{3}\delta_{ij}$ and the values of 1.8 and 0.6 are normally used for the coefficients $c_1$ and $c_2$, respectively. In high Reynolds number isotropic flows, the following treatment is normally applied.

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij} \quad \cdots\cdots (18)$$

Consequently, the algebraic expression for the Reynolds stress is obtained as:

$$\overline{u_i u_j} = k \left[ \frac{2}{3}\delta_{ij} + \frac{(1 - c_2)(P_{ij} - \frac{2}{3}\delta_{ij} P_k)}{(c_1 - 1 + P_k / \varepsilon)\,\varepsilon} \right] . \quad \cdots\cdots (19)$$

Since $P_{ij}$ and $P_k$ consist of the Reynolds stresses and the mean velocity gradients, this algebraic form is implicit in terms of the Reynolds stress.

Therefore, ASM's need to solve the transport equations of $k$ and $\varepsilon$ with successive matrix inversions of the implicit algebraic equation set for the Reynolds stresses.

## 3. Toward a new standard

In this section, the recent novel attempts to improve the EVM's are discussed especially focusing on wall detecting parameters and nonlinear constitutive relations used in the models. However, due to page limitations, unfortunately, many of their model equations are not described, therefore, the referenced papers should be consulted for more details.

### 3. 1 Near-wall modelling

Since the near-wall variation in the local turbulent Reynolds number $R_t$ significantly changes depending on the bulk Reynolds number $Re$ as shown in **Fig. 1**‡, finding another near-wall detector which has general near-wall characteristics has been the main concern of modelling near-wall turbulence.

‡ The results of DNS can be treated as almost exact physics.

In their review, Patel *et al.*[7] emphasized the necessity to have a reasonable near-wall $f_\mu$ distribution because none of their cited models agreed with the data deduced from several different experiments (**Fig. 2**). This triggered many researchers to modify the LRN models, suggesting ways of improving the EVM's. Moreover, since the emergence of the DNS[12-14], a lot of attention has been given to the near-wall asymptotic behaviour of each turbulent quantity because the DNS provided reliable data for every process including unmeasurable correlations.

To obtain a reasonable near-wall distribution of the $f_\mu$ damping function, many recent versions of the LRN $k$-$\varepsilon$ EVM[15-17] have implemented the effects of the dimensionless wall distance (wall unit):

$$y^+ \equiv u_\tau y/\nu \qquad \cdots\cdots (20)$$

where $u_\tau$ and $y$ are the friction velocity and the distance from a wall, respectively. However, one can easily see that none of these LRN EVM's is useful to apply for a flow with a recirculation. In particular, the use of $u_\tau$ is not suitable for such a flow case because it becomes zero at a reattaching point. In this case, the wall unit was sometimes replaced with $R_y\,(\equiv \sqrt{k}\,y/\nu)$. Abe *et al.*[18], however, replaced $u_\tau$ with the Kolmogorov velocity scale, $(\nu\varepsilon)^{1/4}$, and devised the parameter:

$$y^* \equiv (\nu\varepsilon)^{1/4} y/\nu \qquad \cdots\cdots (21)$$

to damp the eddy viscosity in order to obtain reasonable predictions of backward facing step flows. (They later extended the model to a nonlinear $k$-$\varepsilon$ model[19]) The parameter $y^*$ was also used in Kawamura & Kawashima's (KK) LRN $k$-$\varepsilon$ EVM[20].

Nevertheless, the use of the wall distance $y$ limits the model's applicability when considering flow in more complex geometry. The discussion by Lee *et al.*[21] using their DNS results suggested that constructing a universal model depended on identifying dimensionless parameters such as the normalized strain invariant:

$$S \equiv \tau\sqrt{S_{ij}S_{ij}/2} \qquad \cdots\cdots (22)$$

where $\tau$ is the characteristic time scale normally given as the ratio of $k$ and $\varepsilon$, $k/\varepsilon$. In response, in order to obtain the substitute parameter for the wall distance, Yang & Shih[22] (YS) devised the parameter $R$ which consisted of the strain invariant as:

$$R \equiv \frac{k}{\nu\sqrt{S_{ij}S_{ij}/2}} = \frac{k^2/(\nu\varepsilon)}{(k/\varepsilon)\sqrt{S_{ij}S_{ij}/2}} = R_t/S\,. \qquad \cdots\cdots (23)$$

The use of this in the dumping function $f_\mu$ led to good predictive performance in wall shear flows with zero or favorable pressure gradients.

Due to its general characteristics in shear flows as shown in **Fig. 3**, the strain invariant has been recently employed as a near-wall parameter in several other proposals such as Cotton & Ismael[23]

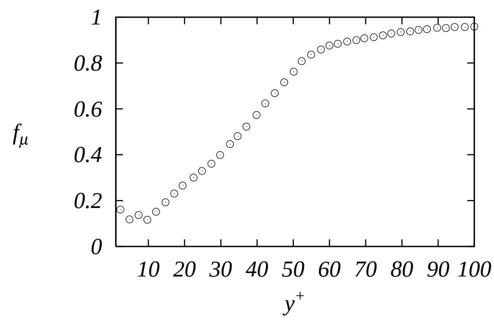


**Fig. 2** Experimentally suggested distribution of the damping function, $f_\mu$ [7].

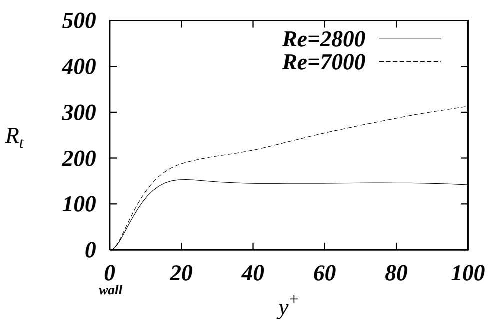


**Fig. 1** Turbulent Reynolds number in plane channel flows by DNS[12,13].

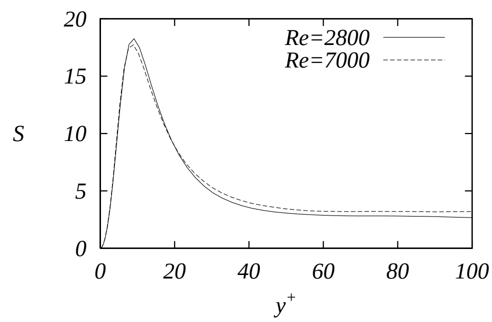


**Fig. 3** Normalized strain invariant in plane channel flows by DNS[12,13].

and the nonlinear $k$-$\varepsilon$ EVM of Craft, Launder & Suga[24,25]. Cotton & Ismael later proposed a $k$-$\varepsilon$-$S$ model[26] coupling with a transport equation for the strain invariant:

$$\frac{DS}{Dt} = D_S + 0.5\,\tau S_{ij}^2 - \frac{S}{\tau}. \qquad \cdots\cdots (24)$$

The transport effects of $S$ gave some reliability in predicting buoyant flows.

Since the desirable variation in the strain invariant as a near-wall parameter is relatively limited near walls, Craft, Launder & Suga[27-29] further introduced the stress invariant $A_2$ ($\equiv a_{ij}a_{ij}$) as another near-wall detector into their nonlinear $k$-$\varepsilon$ EVM. Stress anisotropy is high near a wall and its measure is represented by $A_2$ as shown in **Fig. 4**, hence, $A_2$ can be a near-wall parameter. The value of $A_2$ was obtained by solving its transport equation:

$$\frac{DA_2}{Dt} = -2\frac{A_2}{k}D_k + 2\frac{a_{ij}}{k}D_{ij} - 2\frac{A_2}{k}P_k + 2\frac{a_{ij}}{k}P_{ij} + 2\frac{a_{ij}}{k}\Pi_{ij} + 2\frac{A_2}{k}\varepsilon - 2\frac{a_{ij}}{k}\varepsilon_{ij} \qquad \cdots\cdots (25)$$

with proper models for $\Pi_{ij}$ and $\varepsilon_{ij}$.

Durbin[33] introduced the Reynolds stress component normal to a wall, $\overline{\upsilon^2}$, as a damping parameter for the eddy viscosity of the $k$-$\varepsilon$ EVM as:

$$\nu_t = c_\mu \overline{\upsilon^2}\tau \qquad \cdots\cdots (26)$$

The values of $\overline{\upsilon^2}$ are obtained by solving its modelled transport equation. In fact, the near-wall damping in the eddy viscosity comes from the blocking effect on the fluctuating velocity component normal to a wall-boundary by the existence of the boundary. In this sense, the boundary which gives the blocking effect is not necessarily the wall-boundary. To support this, **Fig. 5** shows a similar damping profile of the eddy viscosity near the free surface ($y/\delta$) of an open channel flow. Hence, directly implementing this effect by the use of $\overline{\upsilon^2}$ in the damping model has a physically correct reason. However, the $k$-$\varepsilon$-$\overline{\upsilon^2}$ model is only applicable in a flow parallel to a wall because $\upsilon$ is not always normal to a wall in complicated geometry. Furthermore, the use of $\overline{\upsilon^2}$ alone in a scalar variable $\nu_t$ leads to severe fundamental inconsistencies since $\overline{\upsilon^2}$ is a component of the Reynolds stress *tensor* and should *not* appear in any *scalar* value.

Thus, it is necessary for a more general eddy viscosity formula to have a physically and mathematically correct damping parameter toward wall or shear-free boundaries. Accordingly, the author noticed the flatness parameter of the Reynolds stress tensor, $A$ ($\equiv 1-\frac{9}{8}(a_{ij}a_{ij} - a_{ij}a_{jk}a_{ki})$), as a damping parameter. He thus extended his work on the $k$-$\varepsilon$-$A_2$ model[27-29] to a $k$-$\varepsilon$-$A$ model[34] by substituting the following $A$-transport equation for the $A_2$ equation.

$$\begin{aligned}\frac{\mathrm{D}A}{\mathrm{D}t} = &-\frac{9}{8k}\left(\frac{3}{2}A_3D_{kk} + 2a_{ij}D_{ij} - 3a_{jk}a_{ki}D_{ij}\right)\\ &-\frac{9}{8k}\left(\frac{3}{2}A_3P_{kk} + 2a_{ij}P_{ij} - 3a_{jk}a_{ki}P_{ij}\right)\\ &-\frac{9}{8k}\left(\frac{3}{2}A_3\Pi_{kk} + 2a_{ij}\Pi_{ij} - 3a_{jk}a_{ki}\Pi_{ij}\right)\\ &+\frac{9}{8k}\left(\frac{3}{2}A_3\varepsilon_{kk} + 2a_{ij}\varepsilon_{ij} - 3a_{jk}a_{ki}\varepsilon_{ij}\right)\end{aligned} \qquad \cdots\cdots (27)$$

Because $A$ is a scalar and vanishes at the wall and shear-free boundaries as shown in **Fig. 6**, its

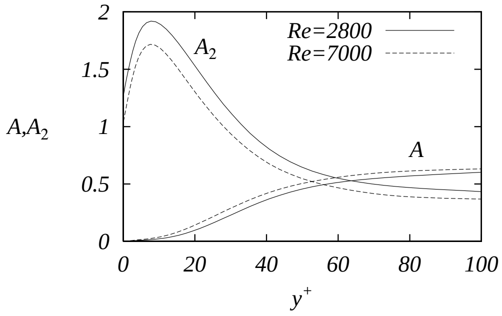


**Fig. 4** Stress invariants in plane channel flows by DNS[12,13].

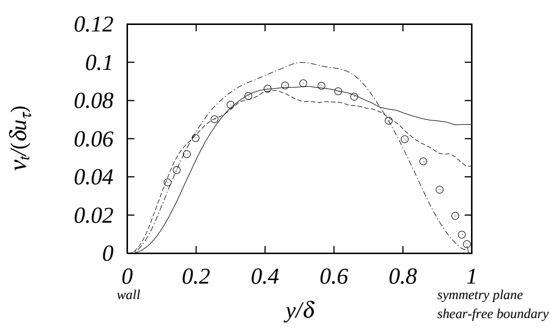


**Fig. 5** Eddy viscosity:
○ ○ ○ , Open-channel[30];
------- , Couette-Poiseuille[32];
——— , Channel ($Re$ =2800)[12];
.......... , Channel ($Re$ =7000)[13].

introduction into the damping function of the eddy viscosity allows one to form a physically and mathematically correct model.

### 3. 2 Nonlinear eddy viscosity modelling

Another important topic in the recent eddy viscosity modelling is a nonlinear extension of the (linear) stress-strain relation, Eq. (3). This approach forms a nonlinear eddy viscosity model (NLEVM). Note that a sort of the NLEVM is sometimes called an explicit ASM due to its optimization process for the coefficients.

The original linear stress-strain relation does not produce meaningful differences between the normal stresses. For example, in shear flows where only $S_{12}$ is nonzero, Eq. (3) leads to isotropic turbulence as:

$$\overline{u_1u_1} = \overline{u_2u_2} = \overline{u_3u_3} = \frac{2}{3}k \qquad \cdots\cdots(28)$$

while the values of the normal stresses are very different from one another in actual flow cases. Thus, the linear model lacks the capability of predicting anisotropic turbulence in many industrially important flows such as turbulence-driven secondary flows, swirling flows, *etc.*

Although the ideas of NLEVM themselves emerged back in the 70's[35,36], until recently, the models of this type were not widely explored. Many attempts at developing and using such schemes have been recently made[37-41]. They all introduced quadratic terms into Eq. (3) as:

$$\begin{aligned} a_{ij} = &- c_\mu \tau S_{ij} + c_1\tau^2 \left( S_{ik}S_{kj} - \frac{1}{3}S_{kl}S_{kl}\delta_{ij} \right) \\ &+ c_2\tau^2 \left( \Omega_{ik}S_{kj} + \Omega_{jk}S_{ki} \right) \\ &+ c_3\tau^2 \left( \Omega_{ik}\Omega_{jk} - \frac{1}{3}\Omega_{lk}\Omega_{lk}\delta_{ij} \right) \end{aligned} \qquad \cdots\cdots(29)$$

where the vorticity tensor, $\Omega_{ij} \equiv \partial U_i / \partial x_j - \partial U_j / \partial x_i$.

The quadratic $c_1$~$c_3$ terms produce discrepancies between the normal stresses. These quadratic NLEVM's thus successfully reproduced turbulence driven secondary flows, however, they did not have sensitivity to streamline curvature (including swirl). Therefore, in order to capture the streamline curvature effects, Craft, Launder & Suga[24] further introduced cubic terms as:

$$\begin{aligned} a_{ij} = &- c_\mu \tau S_{ij} + c_1\tau^2 \left( S_{ik}S_{kj} - \frac{1}{3}S_{kl}S_{kl}\delta_{ij} \right) \\ &+ c_2\tau^2 \left( \Omega_{ik}S_{kj} + \Omega_{jk}S_{ki} \right) \\ &+ c_3\tau^2 \left( \Omega_{ik}\Omega_{jk} - \frac{1}{3}\Omega_{lk}\Omega_{lk}\delta_{ij} \right) \\ &+ c_4\tau^3 \left( S_{ki}\Omega_{lj} + S_{kj}\Omega_{li} \right) S_{kl} \\ &+ c_5\tau^3 \left( \Omega_{il}\Omega_{lm}S_{mj} + S_{il}\Omega_{lm}\Omega_{mj} - \frac{2}{3}S_{lm}\Omega_{mn}\Omega_{nl}\delta_{ij} \right) \\ &+ c_6\tau^3 S_{ij}S_{kl}S_{kl} + c_7\tau^3 S_{ij}\Omega_{kl}\Omega_{kl}. \end{aligned} \qquad \cdots\cdots(30)$$

In fact, the cubic $c_4$~$c_7$ terms have sensitivity to swirl and streamline curvature. They afterwards modified their cubic NLEVM coupling with the effects of $A_2$ to correctly mimic near-wall turbulence[27-29].

Pope[36] showed that the generalized nonlinear stress-strain relation was mathematically equivalent to an explicit form of the ASM. He generalized the nonlinear constitutive relation using the Cayley-Hamilton theorem and solved a matrix obtained by substituting the Reynolds stresses in Eq. (19) with the constitutive relation. Although he outlined the procedure to obtain the coefficients, he was not able to provide the coefficients generally due to the complexity of the algebra. In fact, the generalized constitutive relation includes up to fifth-order products of strain and vorticity tensors. Recently, following Pope's methodology, Taulbee[42] and Gatski & Speziale[43] proposed elaborate coefficients for the three-dimensional flows. Their NLEVM's (explicit ASM's) thus include up to fifth-order terms. However, the roles and necessity of fourth- and fifth-order terms have never been clarified.

Very recently, the author pointed out an inherent defect in the stress-strain relation and tried to remove it. In shear-free turbulence appearing, for example, near the free surface of an open channel flow, all strain and vorticity tensor components vanish. The linear and nonlinear stress-strain relations thus always return isotropic turbulence there (*e.g.*, all the terms on the right hand side of Eq. (30) become 0 in this case) while the actual turbulence is significantly anisotropic. Therefore, the author introduced the following additional term

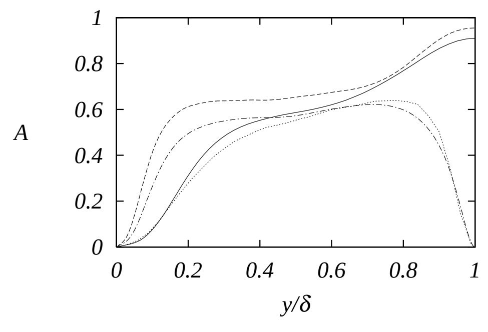


**Fig. 6** Stress flatness parameter:
, Open-channel[31];
, Couette-Poiseuille[32];
, Channel ($Re$ =2800)[12];
, Channel ($Re$ =7000)[13].

$A_{ij}$ composed of the gradient of the stress flatness parameter $A$ into the cubic stress-strain relation: Eq. (30).

$$A_{ij} = c_a' \tau^2 \left( \frac{\partial\sqrt{Ak}}{\partial x_i} \frac{\partial\sqrt{Ak}}{\partial x_j} - \frac{1}{3} \delta_{ij} \frac{\partial\sqrt{Ak}}{\partial x_k} \frac{\partial\sqrt{Ak}}{\partial x_k} \right) \qquad \cdots\cdots (31)$$

Since the distribution of $A$ has a steep gradient near the shear-free boundary as shown in Fig. 6, this additional term does produce anisotropy of turbulence there. The author showed its usefulness for capturing shear-free turbulence combining it with the $k$-$\varepsilon$-$A$ three equation NLEVM[34)]

### 3. 3 Comparisons of model performance

This subsection displays the near-wall performance of typical linear and nonlinear EVM's listed in **Tables 2** and **3**. All the models listed, except for the W92[44)], the SA[45)] and the ARG[46)] models, have already been discussed or referred to. The W92 model is the latest version of the linear $k$-$\omega$ model. Among the nonlinear EVM's, the SA and the ARG models are, respectively, a $k$-$\varepsilon$ model and a $k$-$\omega$ model based on the nonlinear stress-strain model of Gatski & Speziale[43)].

**Fig. 7** compares the predicted turbulent shear stress distributions with the DNS[13)] data near the wall. All the models reproduce the DNS results quite well, though the profile by the W92 model distinctively deviates from the data in the region $y^+ < 20$.

The predicted turbulence energy distributions shown in **Fig. 8**, however, display an interesting fact. The recently proposed nonlinear SA and ARG models predict very similar profiles to that of the rather dated NY model and they are poorer than that of the 24-year-old LS model. Except for them, the recent versions of EVM's have shown quite successful performance. In fact, many of them are in excellent agreement with the DNS.

Since modelling the $\varepsilon$ (or $\omega$) equation is much more difficult than modelling the $k$ equation, thus many of the predicted $\varepsilon$ distributions poorly accord with the DNS data as seen in **Fig. 9**. Nonetheless, the result of the nonlinear CLS model shows quite excellent agreement with the DNS, and those of the linear KK and the nonlinear AKN models are also fairly acceptable.

Another important feature of a LRN model is grid dependency on the predicted results. **Fig. 10** shows the grid dependency on the predictive performance of the mean velocity in the pipe flow measured by

**Table 2** Linear EVM's.

| Model | transport variables | near-wall parameters |
|---|---|---|
| LS: Launder & Sharma(1974)[8)] | $k, \tilde{\varepsilon}$ | $R_t$ |
| W92: Wilcox(1992)[44)] | $k, \omega$ | $R_t$ |
| YS: Yang & Shih(1993)[22)] | $k, \varepsilon$ | $R_t, S$ |
| RM: Rodi & Mansour(1993)[17)] | $k, \varepsilon$ | $R_t, y^+$ |
| KK: Kawamura & Kawashima(1994)[20)] | $k, \tilde{\varepsilon}$ | $R_t, y^*$ |
| CI: Cotton & Ismael(1994)[26)] | $k, \tilde{\varepsilon}, S$ | $R_t, S$ |

**Table 3** Nonlinear EVM's.

| Model | transport variables | near-wall parameters |
|---|---|---|
| NY: Nisizima & Yoshizawa(1987)[38)] | $k, \varepsilon$ | $R_t, y^+$ |
| MK: Myong & Kasagi(1990)[40)] | $k, \varepsilon$ | $R_t, y^+$ |
| SA: Speziale & Abid(1995)[45)] | $k, \varepsilon$ | $R_t, R_y$ |
| ARG: Abid, Rumsey & Gatski(1995)[46)] | $k, \omega$ | $R_t$ |
| AKN: Abe, Kondoh & Nagano(1995)[19)] | $k, \varepsilon$ | $R_t, y^*$ |
| CLS: Craft, Launder & Suga(1995)[29)] | $k, \tilde{\varepsilon}, A_2$ | $R_t, S, A_2$ |
| Suga(1997)[34)] | $k, \tilde{\varepsilon}, A$ | $R_t, A$ |

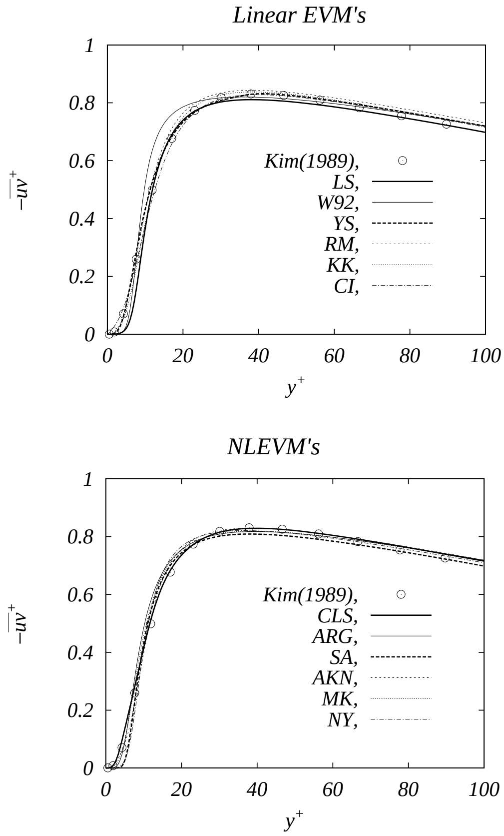


**Fig. 7** Turbulent shear stress.

Laufer[47]. The solid lines noted as 100% are the results with a fine enough grid whose first grid node is located just under unity of the wall unit ($y_1^+ < 1.0$). The lines noted as *x*% are the results using a grid whose grid node density normal to the wall is *x*% of that of the fine enough grid. Obviously, the LS model is very sensitive to the grid density and many of the other LRN models need at least a 50% grid node density of the fine enough grid. (The first grid node's $y^+$ of this 50% grid is about 2.0: $y_1^+$~2.0.) The nonlinear CLS, MK and NY models, however, show equivalent performance even with the 40% grid distributed from $y_1^+$~4.0. On the whole, it can be said the CLS model shows the best performance in the models compared in Fig. 10 in terms of the predictive accuracy and the grid sensitivity.

**Fig. 11-13** show the predicted near-wall turbulent intensities by the NLEVM's compared with the DNS[13] data. The CLS model clearly demonstrates the best performance while the other models do not successfully mimic the stress anisotropy.

The near-wall performance of the author's $k$-$\varepsilon$-$A$ three equation NLEVM is comparable to that of the CLS ($k$-$\varepsilon$-$A_2$) model though it has not been apparently shown. As mentioned in §3.2 and clearly shown in **Fig. 14**, however, the author's $k$-$\varepsilon$-$A$ NLEVM can capture stress anisotropy near the free surface while the CLS model cannot. This model performance of the $k$-$\varepsilon$-$A$ model is believed to be very useful if the model is used to calculate heat and mass transfer through a shear-free interface which is one of the key phenomena of the environmental issues.

## 4. Conclusions

The following aspects may be summarized through this review covering the recent research works on the eddy viscosity modelling of turbulence.

1. Until recently, the wall distance $y$ was often used in the low-Reynolds-number eddy viscosity models, while the use of $y$ limited the model's applicability.

2. Many researchers have started to find general local parameters for detecting wall effects. The proposed near-wall invariant parameters so far are the strain invariant $S$, the stress invariant $A_2$ and the stress flatness parameter $A$. Although they require

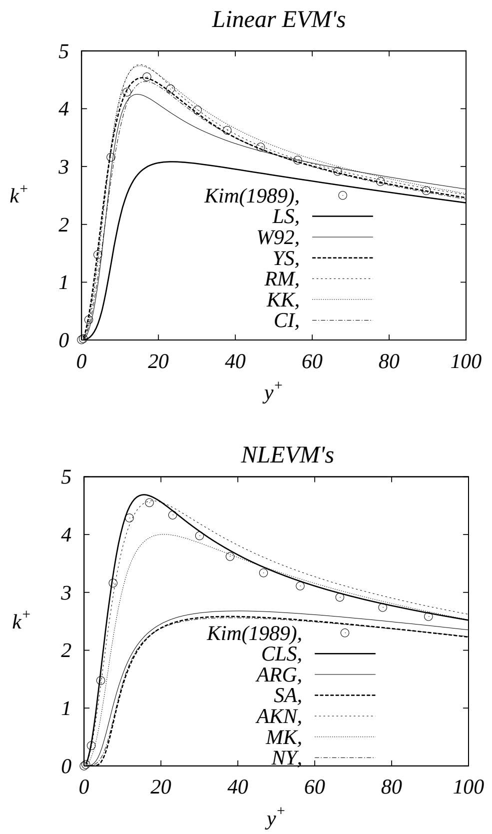


**Fig. 8** Turbulence energy.

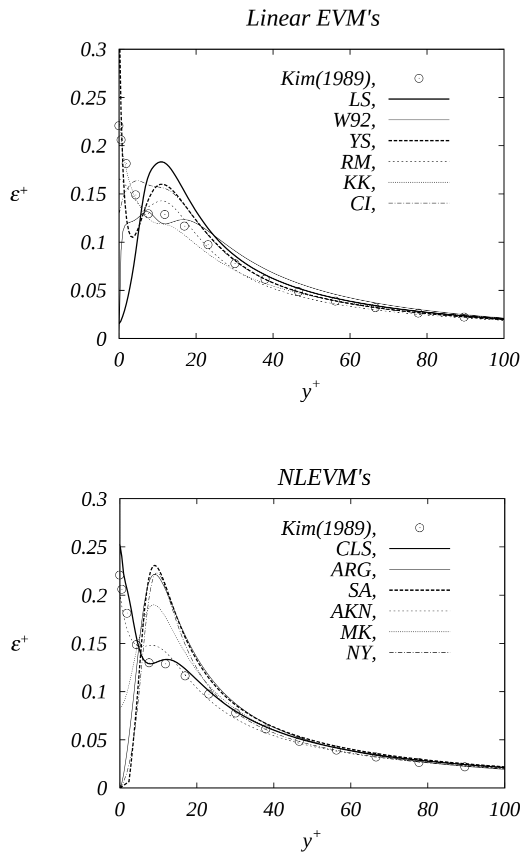


**Fig. 9** Turbulence dissipation rate.

solving their transport equations, the models including their effects showed encouraging results.

3. The use of nonlinear terms in the stress-strain relation is essential to predict complex strain fields. Moreover, the cubic terms are necessary to mimic streamline curvature and swirl effects.

4. The combined effects of the new local near-wall parameters and nonlinear stress-strain relations have significantly extended performance of the eddy viscosity scheme. In particular, the use of *A* has

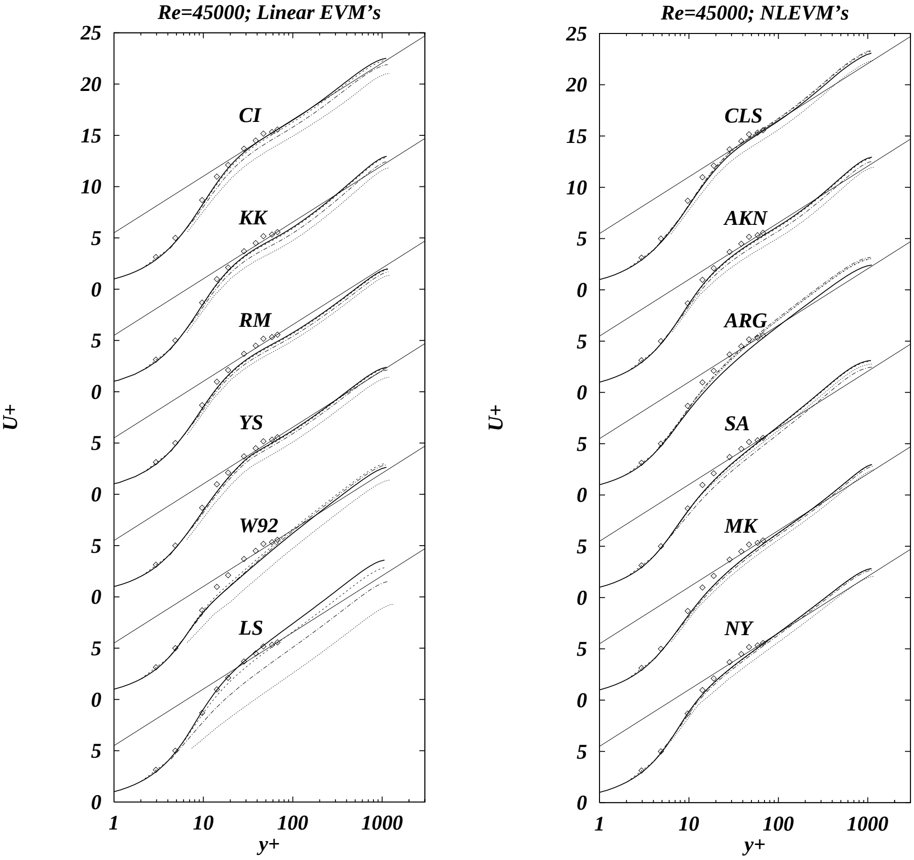


**Fig. 10** Mean velocity distributions:
Symbols, Expt.[47]; ——, 100%; - - - - -, 50%; -·-·-, 40%; ········, 35%.

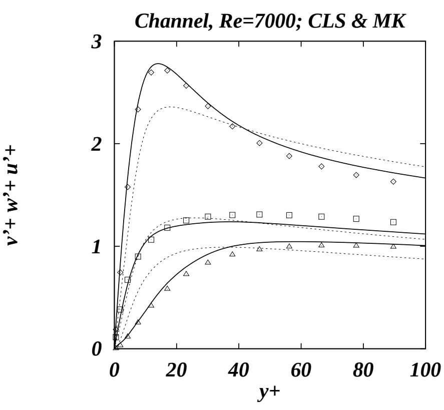


**Fig. 11** Turbulent intensities:
Symbols, DNS[13];
——, CLS[29]; - - - - -, MK[40].

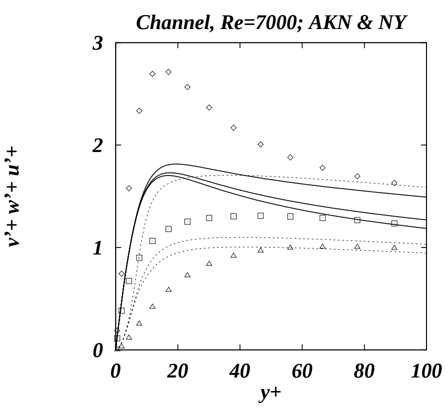


**Fig. 12** Turbulent intensities:
Symbols, DNS[13];
——, AKN[19]; - - - - -, NY[38].

extended the applicability of the eddy viscosity model toward capturing shear-free turbulence which is very important for environmentally oriented issues.

5. Overall, it may be concluded that the nonlinear $k$-$\varepsilon$-$A$ three equation model is the most promising scheme in the eddy viscosity models of turbulence.

In the very near future, the author believes that the standard $k$-$\varepsilon$ model in industrial applications will be replaced with the recently developed low-Reynolds-number nonlinear eddy viscosity models.

To achieve the full potential of the eddy viscosity modelling, however, further attention is expected for optimizing the transport equation for $\varepsilon$. Since the rapid development of DNS will be providing much more detailed data for the modelling, the existing too empirically modelled $\varepsilon$ equations will soon be replaced.

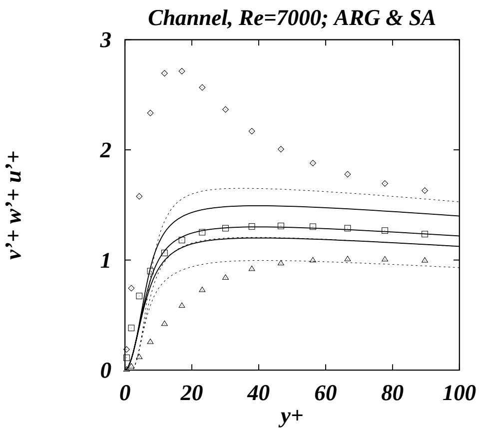


**Fig. 13** Turbulent intensities:
Symbols, DNS[13];
—— , ARG[46]; -------- , SA[45].

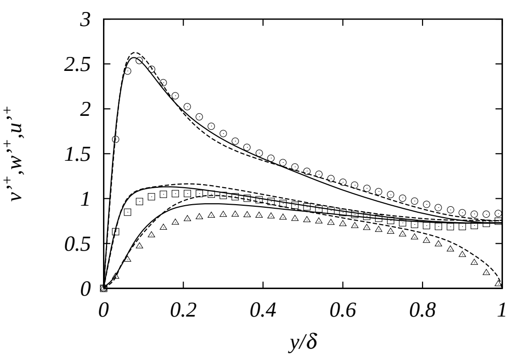


**Fig. 14** Turbulent intensities in an open channel flow:
symbols, DNS[31];
—— , CLS[29]; ------- , $k$-$\varepsilon$-$A$ model[34].

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