Research Report

Linear and Singular Response to Weak Perturbation in Overdamped Systems

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Report received on Jan. 7, 2013

ABSTRACT The extraction of a weak signal in a noisy environment is realized by observing signal-induced modulations. Hence, the investigation of the response to a weak perturbation plays an important role in signal processing research. The response of the system is enlarged by the combination of nonlinearity and an external driving force. The response is generally analyzed by a hybrid of a linear response theory and renormalization theory. In the present paper, we investigate two types of systems: an overdamped system driven by a high-frequency external force, the response of which is simply described in the framework of linear response theory, and an overdamped system with random telegraph noise, the response of which requires a renormalization approach for analysis. We demonstrate that the enhancement of the responses in these systems originates from the bifurcation phenomena.

KEYWORDS Linear Response, Singular Response, Bifurcation, Signal Enhancement, Vibrational Resonance, Stochastic Resonance

1. Introduction

In signal processing problems, such as signal detection and signal estimation, the basic idea in extracting a weak signal from a noisy environment is to identify the signal-induced modulation in the received input.[1-4] The weak signal is detected or estimated only by observing the modulation induced by the weak signal. Therefore, enhancement of the signal-induced modulation is considered to improve the signal processing performance. In information theory, one of the methods by which to identify signal-induced modulation is realized in the sense of probability, i.e., Bayesian estimation.[5-7] Another effort by which to enhance signal-induced modulation is to improve the signal-to-noise ratio (SNR),[8-11] which is more intuitive and easier than Bayesian estimation. It has been reported that the SNR improves when nonlinearity is exploited.[12-20] Therefore, the exploitation of nonlinearity is considered in the presence of noise, while conventional signal processing devices in a noiseless environment are assumed to be linear. Since the enhancement of a weak signal by nonlinearity is translated into a large response to a small perturbation, the response of the system is important in the field of signal processing.

The basic tool for investigating the response to a weak perturbation is linear response theory.[21-23] However, in the conventional linear response theory, the magnitude of the response is assumed to be on the order of the weak perturbation. Therefore, a singular response that differs in order of magnitude from a weak perturbation cannot be analyzed in the framework of the conventional linear response theory. In this case, renormalization theory[24-26] becomes useful for understanding the true behaviors of the system. The method of renormalization bridges the true behaviors of singular response and the naïve theory of linear response.

In the present paper, we investigate the response to a weak perturbation in two types of systems: a system that exhibits a simple linear response and a system that exhibits a singular response. In the next section, we consider the deterministic nonlinear system subjected to a high-frequency driving force. The high-frequency driving force is considered to be the deterministic version of noise. This system is a minimal model, the response of which can be analyzed in the framework of the conventional linear response theory.[17-20, 27] In Section 3, we consider a stochastic system subjected to random telegraph noise.[28-30] Random telegraph noise is two-valued white noise. Owing to this discreteness, the response cannot be understood in the framework of the conventional linear response theory. The response is obtained by coarse-graining of the system states,[31] which is one of major techniques in renormalization theory. Both of the two systems exhibit enhancement of a weak perturbation. The origin of such enhancement is the combination of nonlinearity and driving force. The driving forces are the high-frequency deterministic forces and telegraph noise in Sections 2 and 3, respectively. These systems exhibit bifurcation when the amplitude of the driving force is tuned. The enhancement of the system response is strongly related to bifurcation phenomena.

2. Linear Response: System with Deterministic High-frequency Driving Force

In order to understand the relationship between system response and bifurcation in the linear response regime, we investigate the overdamped deterministic dynamics in this section. We consider the dynamics with external high-frequency driving force(17-20,27) given as

$$\mathrm{d}x/\mathrm{d}t = -\mathrm{d}U(x)/\mathrm{d}x + f(t) + \varepsilon s(t)\ , \quad \cdots\cdots\cdots (1)$$

where $U(x)$ is a potential, $f(t)$ denotes the external force of order unity, the frequency of which is much higher than the inverse of the relaxation time of the system. The small perturbation $\varepsilon s(t)(|\varepsilon| \ll 1)$ and the system state x are regarded as a weak input and output signals, respectively. We assume that $\varepsilon s(t)$ is a low-frequency input, and the corresponding period is much longer than the relaxation time of the system.

The response to a weak perturbation of the system Eq. (1) is understood in the framework of linear response theory. The solution of Eq. (1) is approximately given as

$$x(t) = x_0(t) + \varepsilon \int_{t_i}^{t} \chi(t,\tau) s(\tau)\mathrm{d}\tau + \mathrm{O}(\varepsilon^2)\ , \quad \cdots\cdots (2)$$

where x_0 is the unperturbed solution satisfying $\mathrm{d}x_0/\mathrm{d}t = -\mathrm{d}U(x)/\mathrm{d}x + f(t)$, t_i is the initial time to switch on the perturbation, and $\chi(t_1,t_2)$ denotes the linear response function given as

$$\chi(t_1,t_2) = \exp\left[-\int_{t_2}^{t_1} U''\left(x_0(\tau)\right)\mathrm{d}\tau\right]. \quad \cdots\cdots\cdots (3)$$

Here, we use the notation $U''(x) = \mathrm{d}^2U(x)/\mathrm{d}x^2$. Note that the system response is completely governed by the unperturbed solution in the linear response regime. Since the unperturbed dynamics includes the high-frequency external force and the perturbation is assumed to be low-frequency, the system response is averaged over the time scale of the system relaxation. Correspondingly, after a sufficiently long time, i.e., $t_i \to -\infty$, the response function is asymptotically given as

$$\chi(t_1,t_2) = \exp\left(-\kappa(t_1 - t_2)\right), \quad \cdots\cdots\cdots\cdots\cdots (4)$$

for $t_1 - t_2 > 0$. The exponent κ is given by the time average of U'' as

$$\kappa = \lim_{t\to\infty} \int_{t_i}^{t_i+t} U''\left(x_0(\tau)\right)\mathrm{d}\tau / t. \quad \cdots\cdots\cdots\cdots\cdots (5)$$

The asymptotic expression of the response function Eq. (4) is reproduced from the argument on the time scale of the solution for Eq. (1). Since the weak perturbation is assumed to be slow, the effective dynamics for $x_1 = (x - x_0)/\varepsilon$ is given by reducing the fast oscillation in Eq. (1) as

$$\mathrm{d}x_1/\mathrm{d}t = -\kappa x_1 + s(t). \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (6)$$

Through the derivation of Eq. (6), κ is found to be given as $\kappa = U''_{\mathrm{eff}}(x_{\mathrm{st}})$, where the effective potential $U_{\mathrm{eff}}(x)$ is the time average of the potential $U(x)$ with respect to the fast oscillation, and x_{st} is the stationary solution of the effective dynamics, i.e., $U'_{\mathrm{eff}}(x_{\mathrm{st}}) = 0$. For $U(x) = x^4/4 - x^2/2$ and $f(t) = A\cos\omega t$ as an example, the effective potential is given as $U_{\mathrm{eff}}(x) = x^4/4 - \alpha x^2/2$, where $\alpha = 1 - 3A^2/2\omega^2$. Note that the stationary solution x_{st} exhibits a pitchfork bifurcation when A/ω changes. As shown in the following argument, the bifurcation of the effective slow dynamics is the key to understanding the response in the linear response regime.

Consider a weak sinusoidal perturbation $\varepsilon s(t) = \varepsilon\cos\omega_{\mathrm{s}} t$. Substituting this into Eq. (6) in a straightforward manner yields

$$x_1(t) = Q\cos\left(\omega_{\mathrm{s}} t - \varphi\right), \quad \cdots\cdots\cdots\cdots\cdots\cdots (7)$$

where

$$Q = 1/\sqrt{\kappa^2 + \omega_{\mathrm{s}}^2}, \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (8)$$

and $\varphi = \arctan(\omega_s/\kappa)$. Since κ is given as $\kappa = U''_{eff}(x_{st})$ and the bifurcation of the solution for $U'_{eff}(x_{st}) = 0$ means $U''_{eff}(x_{st}) = 0$, the amplitude of the perturbed solution x_1 reaches a maximum at the bifurcation point. **Figure 1** shows the amplitude Q as a function of A/ω for the bistable potential $U(x) = x^4/4 - x^2/2$. The inset depicts the distance from the bifurcation point $|\alpha|$. As shown in Fig. 1, the amplitude of the perturbed solution has a peak when the amplitude of the additional high-frequency driving force changes. This phenomenon is called vibrational resonance.[17] From our analysis, it is clear that this phenomenon is the result of two external forces of different time scales and the nonlinearity of the system.[27]

3. Nonlinear Response: System with Random Telegraph Noise

In this section, we investigate a system subjected to random telegraph noise. In such a system, the system can exhibit singular behaviors owing to the discreteness of the noise.[28-30] The singularities in the system state distribution can yield a nonlinear response to a weak perturbation.[31] Therefore, a weak input signal, which is a perturbation to the system, can be significantly enhanced.

3. 1 Stationary Properties

We consider an overdamped system subjected to random telegraph noise[31] given as

$$\mathrm{d}x/\mathrm{d}t = -\mathrm{d}U(x)/\mathrm{d}x + \eta , \quad \cdots\cdots (9)$$

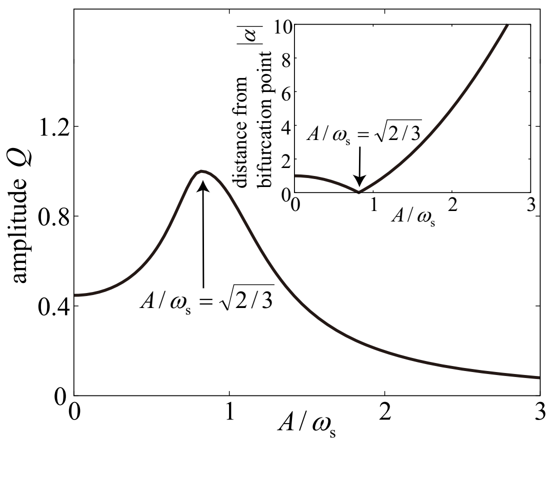

Fig. 1 Amplitude of the output as a function of input amplitude divided by the input frequency. The inset shows the distance from the bifurcation point.

where U denotes a potential. The random telegraph noise η takes two values, $\pm H$. The noise switches as $+H \rightarrow -H$ at switching rate ν_{+-} and $-H \rightarrow +H$ at ν_{-+}. The system state x is regarded as the output from the nonlinear device having a potential of $U(x)$. The stationary probability density of the system state x is obtained by solving the master equation corresponding to the stochastic dynamics Eq. (9). Using the average stationary state $\tilde{x}_{st}$ given by the effective potential $\tilde{U}(x) = U(x) + Hq(\nu_{+-} - \nu_{-+})x/\nu$ as $\tilde{U}'(\tilde{x}_{st}) = 0$ with $\tilde{U}''(\tilde{x}_{st}) > 0$ and $\nu = \nu_{+-} + \nu_{-+}$, the stationary density is expressed as

$$\rho_{st}(x) = \frac{\nu U''(\tilde{x}_{st})\sqrt{\sigma^2/2\pi}}{H^2 - U'^2(x)} \exp\left[-\Phi(x, \tilde{x}_{st})\right], \quad \cdots (10)$$

where $\sigma^2 = 4\nu_{+-}\nu_{-+}H^2/\nu^3 U''(\tilde{x}_{st})$ and

$$\Phi(x_1, x_2) = \nu \int_{x_2}^{x_1} \mathrm{d}x \tilde{U}'(x) / \left[H^2 - U'^2(x)\right]. \quad \cdots\cdots (11)$$

The stationary distribution Eq. (10) has a much richer structure than the Boltzmann density. The density has singularities at x_* satisfying $U'^2(x_*) = H^2$. The number of singularities depends on the potential form and the amplitude of the random telegraph noise.

The stationary densities for three cases are illustrated in **Fig. 2**. Panel (a) shows the asymmetric potential $U(x) = x^2/2 - x^3/3$. For this potential, three types of stationary probability density can be observed if $H < 1/4$. If the amplitude of the telegraph noise exceeds 1/4, the system state starting from $x = 0$ can exceed the potential barrier and become diffuse. For the probability densities illustrated in panels (b) through (d), the initial condition of the system state is chosen as $\delta(x)$, and the amplitude of telegraph noise is set to be $H = 3/16$. The switching rates of telegraph noise (ν_{+-}, ν_{-+}) are set to be $(1, \sqrt{7})$, $(3/8, 3\sqrt{7}/8)$, and (1,1), corresponding to panels (b) through (d). Panel (b) shows the density with compact support. For this switching rate, no divergence appears in the stationary density. Panel (c) shows the stationary density with divergences on the both sides of the boundary, and panel (d) shows the density with divergence on one side of the boundary.

In addition to the stationary probability density, another important stationary property is the stationary probability current. In a bistable potential system, the noise-induced transition from one metastable state to the other can significantly affect the system response. The stationary probability current near the saddle point of the potential gives the escape rate from a metastable state. The escape rate is obtained by the analogy of Kramers' argument as

$$\nu_{\rm esc} = \frac{\left|U''(\tilde{x}_{\rm saddle})U''(\tilde{x}_{\rm st})\right|^{1/2}}{2\pi}\exp\left[-\Phi(\tilde{x}_{\rm saddle},\tilde{x}_{\rm st})\right], \quad \cdots\cdots\cdots\cdots\cdots (12)$$

where $\tilde{x}_{\rm saddle}$ is the saddle point of the effective potential $\tilde{U}'(\tilde{x}_{\rm saddle}) = 0$ with $\tilde{U}''(\tilde{x}_{\rm saddle}) < 0$. The states $\tilde{x}_{\rm st}$ and $\tilde{x}_{\rm saddle}$ are modulated by the external perturbation, and their changes induce the exponential modulation in the escape rate Eq. (12). Therefore, the system can exhibit a singular response, which is the primary goal. A schematic diagram of the potential landscape is shown in **Fig. 3**.

In **Fig. 4**, we compare the results of the analytical calculation for the escape rate with numerical simulations. The potential is taken as $U(x) = x^4/4 - x^2/2$. Good agreement is achieved for moderately large ν/H^2. For comparison, we show the results for thermal activation with effective temperature $k_B T_{\rm eff} = H^2/\nu$ given by the conventional Kramers' theory.(32) In order to ensure the validity of this comparison, symmetric telegraph noise $\nu_{+-} = \nu_{-+}$ has been used. The amplitude of telegraph noise is set to be $H = 0.5$. Figure 4 shows that the exponent of the escape rate for telegraph noise-driven system is clearly different from the Gaussian noise-driven (thermally activated) system.

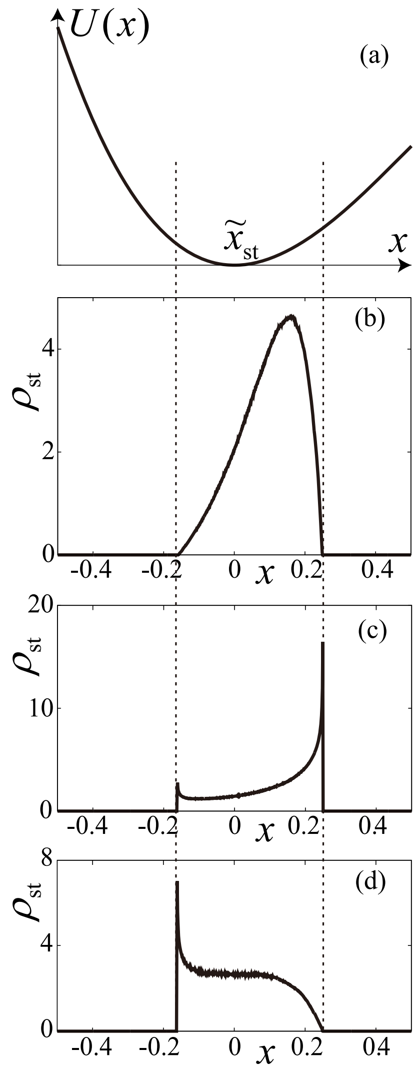

Fig. 2 Three types of stationary probability density trapped in a metastable state (b)-(d). Panel (a) shows the corresponding potential landscape.

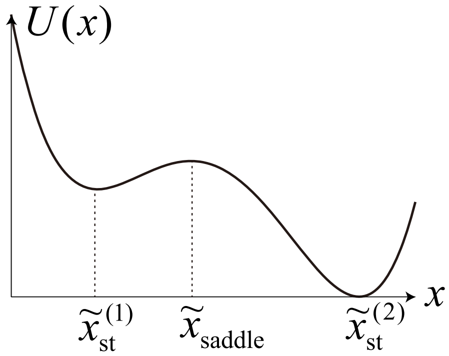

Fig. 3 Schematic image of the potential landscape.

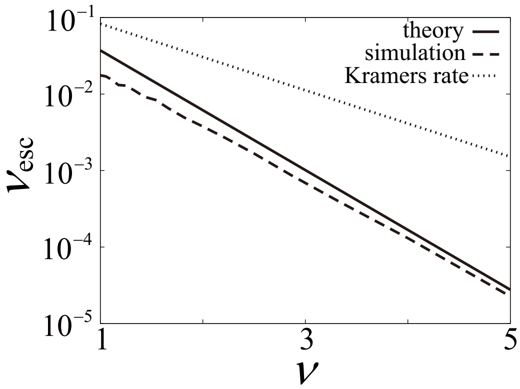

Fig. 4 Escape rates obtained theoretically (solid line) and by numerical simulation (broken line). The dotted line shows the Kramers' rate for the corresponding thermal activation.

3. 2 Response to Weak Periodic Perturbation

The response of the system Eq. (9) to a weak periodic perturbation is described by a hybrid use of linear and singular response functions. Consider that the system is governed by a metastable potential. In this case, the modulation of the system state can be decomposed into two parts. One is the intrawell modulation, which is small and thus described by the conventional linear response theory, and the other is the interwell modulation that yields the singular response.

The linear response is described by the conventional response function

$$\tilde{\chi}(\omega)=\left[U''(\tilde{x}_{\rm st})-i\omega\right]^{-1}. \quad (13)$$

The solution for the perturbed system

$$\mathrm{d}y/\mathrm{d}t=-\mathrm{d}U(y)/\mathrm{d}y+\eta+A\cos\omega_{\rm s}t, \quad (14)$$

is modulated as

$$\langle\delta x(t)\rangle=A\,\mathrm{Re}\left[\tilde{\chi}(\omega)\exp(-i\omega_{\rm s}t)\right]/2, \quad (15)$$

where $\delta x(t)$ is the difference between the solutions for the perturbed dynamics Eq. (14) and the unperturbed dynamics Eq. (9), $\delta x(t) = y(t) - x(t)$. The notation $\langle\cdots\rangle$ denotes the expectation value.

The singular response is given by the modulation in the escape rate Eq. (12). Since the escape rate is modulated by the periodic perturbation, the rates of the interwell transitions change periodically. Such transition rates are given as

$$R_{nm}(t)=r_{nm}\left(1-A\phi_n\cos\omega_{\rm s}t\right), \quad (16)$$

where R_{nm} denotes the transition rate from well n to well m, r_{nm} is the unperturbed rate of escape from well n given as

$$r_{nm}=\frac{\left|U''(\tilde{x}_{\rm st}^{(n)})U''(\tilde{x}_{\rm saddle})\right|^{1/2}}{2\pi}\exp\left[-\Phi(\tilde{x}_{\rm saddle},\tilde{x}_{\rm st}^{(n)})\right], \quad (17)$$

and ϕ_n is expressed using a functional derivative as

$$\phi_n=-\int_{\tilde{x}_{st}^{(n)}}^{\tilde{x}_{\rm saddle}}\frac{\delta\Phi(\tilde{x}_{\rm saddle},\tilde{x}_{\rm st}^{(n)})}{\delta U'(x)}\mathrm{d}x \quad (18)$$

Here, $x_{\rm st}^{(n)}$ denotes the metastable state in well n. The interwell transition rate R_{nm} governs the population in each well P_n. The temporal change of the population is well described by the population dynamics as

$$\mathrm{d}P_n/\mathrm{d}t=\sum_{m(\neq n)}\left[R_{mn}(t)P_m-R_{nm}(t)P_n\right], \quad (19)$$

with normalization condition $\Sigma_n P_n = 1$. The population dynamics Eq. (19) with the interwell transition rate Eq. (16) yields the periodic modulation in the population. Since the interwell interval is significantly larger than the amplitude of the perturbation A, the interwell transition yields the singular response to a weak perturbation.

The intrawell modulation around each local minimum of the effective potential $\tilde{x}_{\rm st}^{(n)}$ is given by Eq. (15). The population of each local minimum $\tilde{x}_{\rm st}^{(n)}$ is given by P_n. Therefore, the response to a weak periodic perturbation is given by the hybrid use of intrawell and interwell responses as

$$\sum_n\left[x_{\rm st}^{(n)}+A\,\mathrm{Re}\,\tilde{\chi}_n(\omega_{\rm s})\exp(-i\omega_{\rm s}t)/2\right]P_n(t), \quad (20)$$

where $\tilde{\chi}_n(\omega)=\left[U''(x_{\rm st}^{(n)})-i\omega\right]$.

For the bistable potential system $U(x) = x^4/4 - x^2/2$, the exponent of the escape rate $\Phi(\tilde{x}_{\rm saddle},\tilde{x}_{\rm st})$ can be written as $\Phi(\tilde{x}_{\rm saddle},\tilde{x}_{\rm st})=\nu S(H)$. The function $S(H)$ is plotted in **Fig. 5**. The function $S(H)$ diverges for $H \to 2/3^{3/2} \approx 0.385$. This means that the singular response becomes relevant when the amplitude of random telegraph noise is set to be near $H = 2/3^{3/2}$. Such a nonmonotonic change in the magnitude of the system response with the monotonic change of noise amplitude is known as stochastic resonance.(12,13) The most commonly investigated stochastic resonance is induced by Gaussian noise.(12-16) For Gaussian noise, the response is not singular. Systems with telegraph noise can exhibit an unconventional stochastic resonance, which yields a strongly singular response.(33,34)

Note that the stationary density Eq. (10) bifurcates [35] at $H = 2/3^{3/2}$ as the amplitude of the telegraph noise changes. For $H < 2/3^{3/2}$, the stationary density is trapped in a potential well, and the state cannot exceed the potential barrier. For $H > 2/3^{3/2}$, on the other hand, the stationary density spreads over all potential wells. Therefore, the enhancement of the system response again originates from the bifurcation, as well as the deterministic system investigated in Section 2.

Figure 6 shows the response to a weak periodic perturbation $A\cos\omega_s t$ in a bistable potential system $U(x) = x^4/4 - x^2/2$. The switching rate of telegraph noise is symmetric as $\nu_{+-} = \nu_{-+} = 1.0$. Since the unperturbed solution gives $\langle x(t)\rangle = 0$, $\langle y(t)\rangle$ is plotted in order to show the response of the system. The amplitude of random telegraph noise is taken as $H = 0.384$, which is close to the critical value, i.e., $H = 2/3^{3/2}$. The parameters for the perturbation are taken as $A = 0.1$ and $\omega_s = 10^{-3}$. In Fig. 6, the theoretical result Eq. (20) (solid line) is compared with the numerical simulation result (broken line). The strong square-wave-like form of $\langle y(t)\rangle$ originates from the relevance of the interwell transition.

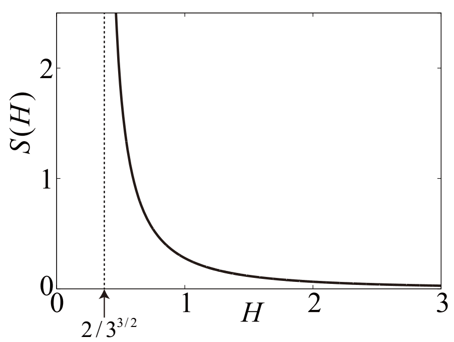

Fig. 5 Scale function for the exponent of the escape rate.

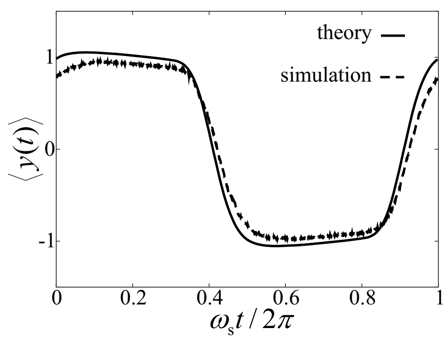

Fig. 6 Response to a weak sinusoidal perturbation.

4. Conclusion

In this paper, we have investigated the response to a weak perturbation in nonlinear systems. It has been demonstrated that the response is enlarged owing to the bifurcation induced by tuning the amplitude of the external driving force in nonlinear systems. Using such systems as devices for signal processing, a large SNR can be obtained. In a stochastic system, however, the large SNR does not ensure accurate signal estimation in a deterministic sense. The signal estimation should be carried out in a statistical sense: a number of samples of received input are required. The large SNR requires fewer samples in order to obtain a statistically relevant estimation. Therefore, systems exhibiting large SNRs can be exploited as devices with short signal processing times.

Most conventional signal processing devices are linear. The nonlinearity of the device is avoided in designing devices. However, as demonstrated herein, the significant enhancement of a weak signal is relevant in nonlinear systems. In particular, the nonlinear response shown in Section 3 cannot appear in linear systems. Therefore, the nonlinearity is attractive in the field of signal processing. This conclusion suggests a device design approach opposite that of the currently accepted approach. The analysis presented herein will be key in designing signal processing devices in the future.

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